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MORSE THEORY FOR SYMMETRIC FUNCTIONALS ON THE SPHERE  
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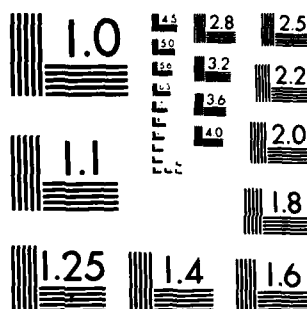
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MORSE THEORY FOR SYMMETRIC FUNCTIONALS  
ON THE SPHERE AND AN  
APPLICATION TO A BIFURCATION PROBLEM

Vieri Benci and Filomena Pacella

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MORSE THEORY FOR SYMMETRIC FUNCTIONALS ON THE SPHERE  
AND AN APPLICATION TO A BIFURCATION PROBLEM

Vieri Benci<sup>1</sup> and Filomena Pacella<sup>2</sup>

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ABSTRACT

In this paper we use Conley's index to study the critical points of a functional  $f$  on a finite dimensional sphere in presence of a symmetry group.

We prove a theorem which leads to a lower bound on the number of critical points of  $f$  when the group is finite, even if the action is not free.

This investigation has been motivated by the following bifurcation problem:

$$Au = \lambda u$$

where  $A$  is a variational  $G$ -equivariant operator.

We give an estimate on the number of "branches" bifurcating from an eigenvalue of  $A'(0)$ .

AMS (MOS) Subject Classifications: 35A15, 35B32, 58E05, 57S17.

Key Words: Morse Theory, Group actions, Variational operators, Bifurcation.

Work Unit Number 1 - Applied Analysis.

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# SIGNIFICANCE AND EXPLANATION

We consider the bifurcation problem

$$A(u) = \lambda u$$

where  $A$  is a nonlinear-variational operator with  $A(0) = 0$ . For such operators it is well known that every eigenvalue  $\lambda_0$  of the linearized operator  $A'(0)$  is a bifurcation point. If the problem exhibits some symmetry, the eigenvalues of  $A'(0)$  are generally degenerate.

Under suitable assumptions, we prove that the number of "branches" which bifurcate from  $(0, \lambda_0)$  is larger than or equal to the multiplicity of  $\lambda_0$ . This very concrete problem leads us to the study of symmetric functionals on the  $n$ -dimensional sphere which we analyze using the Conley index.

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MORSE THEORY FOR SYMMETRIC FUNCTIONALS ON THE SPHERE  
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Vieri Benci<sup>1</sup> and Filomena Pacella<sup>2</sup>

INTRODUCTION

We consider the following nonlinear eigenvalue problem

$$(0.1) \quad A(u) = \lambda u$$

where  $A$  is an operator of class  $C^1$  defined on a real Hilbert space  $H$  such that  $A(0) = 0$ .

We suppose that  $A$  is variational i.e. there exists a functional  $J : H \rightarrow \mathbb{R}$  such that:

$$(0.2) \quad A = J'$$

where  $J'$  denotes the gradient of  $J$ . It is well known (cf [K], [Bo], [M]) that, under these assumptions, every finite multiplicity isolated eigenvalue of the linearized equation:

$$(0.3) \quad A'(0)v = \lambda v$$

is a bifurcation point of  $A$ .

However, if  $A$  is odd and  $\lambda_0$  is an isolated eigenvalue of (0.3) with multiplicity  $n$ , Böhm and Marino ([Bo],[M]) have proved that at least  $2n$  "branches" of solutions of (0.1) bifurcate from  $\lambda_0$ , i.e. the multiplicity of the linearized problem persists in the nonlinear problem (at least in a certain sense).

In this case the functional  $J$  is invariant with respect to the action of the group  $G_0 = \{Id, -Id\} \cong \mathbb{Z}_2$ .

Moreover the eigenvalue problem can be reduced, by virtue of the procedure of Böhm and Marino, to the study of the critical points of  $J$  on a manifold isomorphic to the

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(n-1)-dimensional sphere  $S^{n-1}$ . The result finally follows from the fact that

$$\text{cat}(S^{n-1}/G_0) = n$$

where  $\text{cat}(\cdot)$  denotes the Ljusternik-Schnirelman category.

The purpose of this paper is to investigate if there are other group actions which lead to a multiplicity result of the Böhme-Marino type, even in the case when the L.-S. category of the quotient space is not known and when the action of the group is not free (cf. sect. 1).

In theorem 2.1 we obtain an estimate of the number of critical points of a functional defined on  $S^{n-1}$  which is invariant with respect to the action of a finite group. This result allows us to extend the Böhme-Marino result to more general situations (Th. 3.1).

Since in our situation we cannot use the L. S. category, the choice of the Morse theory seems suitable. We have also used the Conley index to treat the degenerate case.

# 1. Notation and preliminaries

Let  $M$  be a  $n$ -dimensional compact manifold and  $f$  a  $C^1$  function on  $M$ . A point  $x_0$  in  $M$  is a critical point for  $f$  if  $f'(x_0) = 0$ . Let  $K$  be the set of the critical points of  $f$  and we suppose that  $K$  has a finite number of connected components. The collection of its connected components will be denoted by  $\{M_\pi\}_{\pi \in P}$ .

Since the sets  $M_\pi$  are isolated in  $K$ , using the values of  $f$ , we can order them in such a way that they form a Morse decomposition in the sense of [C] and [CZ]. Then to each  $M_\pi$  there corresponds a formal polynomial  $p(\cdot, M_\pi)$  which expresses the "Conley's index" of  $M_\pi$  in the gradient flow:

$$(1.2) \quad \dot{x} = -f'(x)$$

For instance, if  $f \in C^2$  and  $M_\pi = \{x_\pi\}$ , where  $x_\pi$  is a nondegenerate critical point, then:

$$(1.3) \quad p(t, x_\pi) = t^d$$

where  $d$  is the number of negative eigenvalues of the Hessian  $Hf$  in the point  $x_\pi$ . If  $f \in C^2$  and  $M_\pi$  is given by an isolated, degenerate critical point  $x_\pi$ , then, in general,  $p(t, x_\pi)$  is not equal to  $t^d$  (see [C] for some examples).

We give the following definition:

**DEFINITION 1.1** - The number  $\mu(M_\pi) = p(1, M_\pi)$  is said to be the multiplicity of  $M_\pi$ .

From our definition it follows that each nondegenerate critical point has multiplicity 1.

Definition 1.1 is justified by the following argument. It is known (see also [MP]) that if  $f$  is  $C^2$  and has some degenerate critical points, then you can perturb it obtaining another function  $\tilde{f}$ , "near"  $f$ , which has only nondegenerate critical points. By using the continuity property of Conley's index this implies that if  $x_0$  is an isolated degenerate critical point of  $f$ , with  $\mu(x_0) = p$ , then it splits in at least  $p$  nondegenerate critical points of  $\tilde{f}$ .

To the manifold  $M$  there corresponds a formal polynomial  $P(t, M)$ , that is the "Poincaré" polynomial of  $M$  which "represents" the cohomology of  $M$  with coefficients in some field  $F$  (see [Sp]). For instance if  $M = S^n$ , then  $P(t, S^n) = 1+t^n$ , for each field  $F$ , because the cohomology groups of  $S^n$  are:



$$H^i(S^n; F) = \begin{cases} F & \text{for } i = 0 \text{ or } n \\ 0 & \text{otherwise} \end{cases}.$$

With this understood, the "generalized" Morse relations are:

$$(1.4) \quad \sum_{\pi} p(t, M_{\pi}) = P(t, M) + (1+t)Q_f(t)$$

where  $Q_f(t)$  is a polynomial with nonnegative coefficients which depends on  $f$ . Because of this we have:

$$(1.5) \quad \sum_{\pi} p(t, M_{\pi}) \geq P(t, M)$$

where this inequality must be interpreted in the sense that every coefficient of the left hand side polynomial is greater than or equal to the corresponding coefficient of  $P(t, M)$ .

In particular, if  $f \in C^2$  is nondegenerate we have:

$$(1.6) \quad \sum_{\pi} p(t, M_{\pi}) = \sum_{i=0}^m \alpha_i t^i$$

where  $\alpha_i$  is the number of critical points whose Morse index is equal to  $i$ . Then, since the coefficients of  $P(t, M)$  are the "Betti" numbers of  $M$ , from (1.4) and (1.6) we obtain the classical Morse inequalities for  $f$ .

REMARK 1.1 - The definition of Conley's index applies in much more general situations than that described here. The only thing that you require in order to define this index is the presence of a flow in a topological space (see also [SM]).

We end this section recalling something about group actions and introducing some notations (we refer to [Br] for proofs and details).

Let  $G$  be a compact Lie-group and  $X$  a topological space.

An action of  $G$  on  $X$  is a map

$$\phi : G \times X \rightarrow X, \quad \phi(g, x) = gx$$

with the following properties:

$$(1.7) \quad 1x = x \text{ for each } x \in X, 1 \in G$$

$$(1.8) \quad g_1(g_2x) = (g_1g_2)x \quad g_1, g_2 \in G, x \in X.$$

A space with an action of a group  $G$  is called a  $G$ -space. We say that the action of  $G$  on  $X$  is free, if:

$$(1.9) \quad g \neq 1 \implies gx \neq x \text{ for each } x \in X.$$

We denote by  $O_x = \{gx, g \in G\}$  the orbit of  $x$  and by  $X/G$  the set of all orbits. When the action is free, each  $O_x$  looks like  $G$  in the sense that there exists a natural homeomorphism between  $O_x$  and  $G$ .

The closed subgroup of  $G$  defined by

$$G_x = \{g \in G : gx = x\}$$

is called the isotropy group of  $x$ . Of course, if the action is free  $G_x = \{1\}$ , for each  $x$ .

If  $G_x = G$ , we say that  $x$  is a fixed point under the action of  $G$ . We want to point out that if  $X$  is a manifold and the action of  $G$  is not free, then, in general,  $X/G$  is not a manifold.

Given the function  $f : X \rightarrow \mathbb{R}$ , we say that  $f$  is  $G$ -invariant if  $f(gx) = f(x)$  for each  $x \in X$  and  $g \in G$ . If  $X$  and  $Y$  are two  $G$ -spaces, we say that a function  $F : X \rightarrow Y$  is  $G$ -equivariant if:

$$F(gx) = gF(x) \text{ for each } x \in X \text{ and } g \in G.$$

Finally we recall that, if  $H$  is a real Hilbert space, the action of a group  $G$  on  $H$  is said to be orthogonal if  $(x, y) = (gx, gy)$ , for every  $g \in G, x, y \in H$ , where  $(\cdot, \cdot)$  is the scalar product in  $H$ . From now on everytime we say "orthogonal group" it is understood that we are considering a group with an orthogonal action.

## 2. Critical points of symmetric functions on the sphere.

Let  $S^n$  denote the sphere in  $\mathbb{R}^{n+1}$ ,  $G$  a finite group acting orthogonally on  $S^n$  and let  $f : S^n \rightarrow \mathbb{R}$  be a  $G$ -invariant  $C^1$ -function.

We denote by  $|G|$  and  $|G_x|$  the order of  $G$  and  $G_x$  respectively and by  $|O_x|$  the number of distinct elements of  $O_x$ . We set

$$\delta = \text{G.C.D.}\{|O_x| : x \in K\}$$

where  $K$  is the set of critical points of  $f$  and G.C.D. denotes the greatest common divisor.

REMARK 2.1 - We observe that  $|O_x| = \frac{|G|}{|G_x|}$ , thus  $|O_x|$  is a divisor of  $|G|$ . If the action is free, then  $\delta = |G|$ .

The main result of this section is the following theorem:

THEOREM 2.1: If  $\delta > 2$  then  $f$  has at least  $n+1$  orbits of critical points provided that each critical point is counted with its multiplicity in the sense of Definition 1.1.

Proof: If  $f$  has infinitely many critical points then the theorem is proved. So we can suppose that  $f$  has only finitely many critical points. This implies that each of them is a connected component in the set  $K$ . Hence the sets  $M_\pi$  of the Morse decomposition defined in the previous section are just points and the generalized Morse equalities (1.4) are:

$$(2.2) \quad \sum_{\pi} p(t, M_\pi) = P(t, S^n) + (1+t)Q_f(t) .$$

Writing explicitly (2.2) we have:

$$(2.3) \quad \sum_{i=0}^n \alpha_i t^i = 1+t^n + (1+t) \sum_{i=0}^{n-1} \beta_i t^i$$

where  $1+t^n$  represents the cohomology of  $S^n$  with a field of coefficients  $F$ .

Note that the maximum exponent in the left hand side of (2.3) is  $n$  because the index-polynomial of each critical point cannot contain a power bigger than  $t^n$ , if  $n$  is the dimension of the manifold on which  $f$  is defined.

Since  $f$  is invariant under the action of  $G$  the polynomial  $p(t, M_\pi)$  is invariant too, i.e. it is the same for points in the same orbit. This implies that each coefficient

$\alpha_1$  different from 0 is a multiple of  $\delta$ . From (2.3) it follows that

$$(2.4) \quad \begin{cases} \alpha_0 = 1 + \beta_0 \\ \alpha_1 = \beta_0 + \beta_1 \\ \vdots \\ \alpha_i = \beta_{i-1} + \beta_i \\ \vdots \\ \alpha_n = 1 + \beta_{n-1} \end{cases}$$

We want to prove that each  $\beta_i$  is different from 0 and is not a multiple of  $\delta$ , arguing by induction. For  $\beta_0$  this is true since  $\alpha_0$  is a multiple of  $\delta$ .

Now suppose that this is true for  $\beta_{i-1}$ . Then (2.4) implies  $\alpha_i \neq 0$  and so  $\alpha_i = p\delta$ , for some  $p \neq 0$ . Therefore, since  $\beta_{i-1}$  is not a multiple of  $\delta$ ,  $\beta_i$  is different from 0 and is not a multiple of  $\delta$ . Having proved that  $\beta_i > 0$ , for each  $0 \leq i \leq n-1$ , the assertion follows from (2.4), because each  $\alpha_i$  has to be greater than zero. □

REMARK 2.2 - From the proof of the previous theorem it turns out that since each  $\alpha_i$  is different from 0, for every  $0 \leq i \leq n$ ,  $f$  has at least  $\delta$  critical points whose polynomials contain the power  $t^i$ .

COROLLARY 2.1 - If  $f \in C^2$  is not degenerate, then there exists at least one orbit of  $G$  consisting of critical points of  $f$  with Morse index  $i$ , for  $i = 0, \dots, n$ . In particular  $f$  has at least  $n+1$  critical points which lie in different orbits.

PROOF: It follows from remark 2.2 and the fact that for each nondegenerate critical point  $x_\pi$ ,  $p(t, \{x_\pi\}) = t^d$  where  $d$  is the Morse index of  $x_\pi$ . □

COROLLARY 2.2 - If the action of  $G$  on  $S^n$  is free, then  $f$  has at least  $(n+1)|G|$  critical points.

REMARK 2.3 - If  $G = \mathbb{Z}_p$  and the action is free the assertion of Corollary 2.1 can be obtained using the Ljusternik-Schnirelman category. In fact, consider the diagram

$$\begin{array}{ccc}
 S^n & & \\
 \pi \downarrow & \searrow f & \\
 S^n/\mathbb{Z}_p & \xrightarrow{\tilde{f}} & \mathbb{R}
 \end{array}$$

where  $\pi$  is the projection and  $\tilde{f}$  is defined by the diagram itself. Since  $\text{cat}(S^n/\mathbb{Z}_p) = n+1$ , then  $\tilde{f}$  has at least  $n+1$  critical points. Then  $f$  has at least  $(n+1)|G|$  critical points. Under this point of view Corollary 2.1 is an extension of this result any finite group  $G$ . However the knowledge of  $\text{cat}(S^n/G)$  gives extra information in "degenerate" case. In fact if  $\text{cat}(S^n/G) = n+1$ , there exist always at least  $(n+1)|G|$  distinct critical points, no matter how degenerate they are. If  $\text{cat}(S^n/G) < n$ , then using Corollary 2.2, we can conclude that there exists at least  $(n+1)|G|$  critical points only if they are counted with their multiplicity in the sense of definition 1.1.

**REMARK 2.4** - The result of Theorem 2.1 is of topological nature. Therefore the same result applies not only to the  $n$ -dimensional sphere but to any manifold  $G$ -homeomorphic to it.

### 3. A bifurcation theorem.

We begin this section by stating the following Lemma which is the equivariant version of a Lemma of A. Marino [M] (cf. also [Bo]).

LEMMA 3.1 - Let  $G$  be a group acting orthogonally on the Hilbert spaces  $X_1, X_2$  and  $Y$ .  
Let  $\Omega$  be a  $G$ -invariant neighborhood of  $0$  in  $X_1 \times X_2^{(1)}$  and set  
 $\Gamma = \{(x_1, x_2) \in X_1 \times X_2 : |x_2| < \gamma |x_1|, \gamma > 0\}$ . Let  $B : \Gamma \cap \Omega - \{0\} \rightarrow Y$  be a  $C^1$ -operator  
with the following properties:

- i)  $\lim_{x_1 \rightarrow 0} \frac{B(x_1, 0)}{|x_1|} = 0$
- ii) there exists an isomorphism  $B_0$  from  $X_2$  to  $Y$  such that:  
 $B_{x_2}^1(x_1, x_2) \rightarrow B_0$  strongly if:  
 $|x_1| + |x_2| \rightarrow 0$  and  $\frac{|x_2|}{|x_1|} \rightarrow 0$
- iii)  $B_{x_1}^1(x_1, x_2) \rightarrow 0$  strongly if  $|x_1| + |x_2| \rightarrow 0$  and  $\frac{|x_2|}{|x_1|} \rightarrow 0$
- iv)  $B$  is  $G$ -equivariant.

For  $c, \delta > 0$  set

$$C = \{(x_1, x_2) \in X_1 \times X_2 : |x_2| < c |x_1|\}$$

$$F = \{x_1 \in X_1 : 0 < |x_1| < \delta\}.$$

Then

- (I)  $\left\{ \begin{array}{l} \text{If } c \text{ and } \delta \text{ are sufficiently small there exists a unique function} \\ \phi : F \rightarrow \Omega \\ \text{with its graph in } C \text{ such that} \\ B(x_1, x_2) = 0 \text{ if and only if } x_2 = \phi(x_1) \\ \text{for every } (x_1, x_2) \in C \cap \Omega. \end{array} \right.$
- (II) The function  $\phi$  is  $C^1$  and  $\lim_{x_1 \rightarrow 0} \phi^1(x_1) = 0$

(?)

In  $X_1 \times X_2$  there is the "diagonal" action of  $G$ :

$$g(x_1, x_2) = (gx_1, gx_2) \quad g \in G, (x_1, x_2) \in X_1 \times X_2.$$

(III)  $\phi$  is  $G$ -equivariant.

PROOF: The proof of (I) and (II) is contained in [M]. We have only to prove that  $\phi$  is  $G$ -equivariant. For every  $g \in G$  and  $x_1 \in F$ , by (I),  $\phi(gx_1)$  is the only point such that

$$(3.1) \quad B(gx_1, \phi(gx_1)) = 0.$$

But we also have

$$B(gx_1, g\phi(x_1)) = gB(x_1, \phi(x_1)) = 0.$$

By the above equality and (3.1) it follows that  $\phi(gx_1) = g\phi(x_1)$ . The proof is complete.  $\square$

Now let  $H$  be a Hilbert space on the real field with the orthogonal action of a group  $G$  and let  $\Omega \subseteq H$  be a  $G$ -invariant open set, with  $0 \in \Omega$ .

We consider the equation:

$$(3.2) \quad A(u) = \lambda u$$

where  $A$  is a variational  $G$ -equivariant operator of class  $C^1$ , i.e. there exists a  $G$ -invariant functional of class  $C^2$ ,  $J : H \rightarrow \mathbb{R}$  such that

$$(3.3) \quad A(u) = J'(u)$$

where  $J'$  denotes the gradient of  $J$ . Moreover, we suppose that  $A(0) = 0$  so that  $0$  provides the trivial solution of (3.2).

Now let's suppose that  $\lambda_0$  is an isolated eigenvalue of  $A'(0)$  of finite multiplicity. By our assumptions  $A'(0)$  is a selfadjoint operator, then we have the following splitting of  $H$ :

$$H = H_1 \oplus H_2$$

where  $H_1 = \ker(A'(0) - \lambda_0 I)$  ( $I$  denotes the identity in  $H$ ) and  $H_2 = H_1^\perp = \text{range}(A'(0) - \lambda_0 I)$ .  $P_1$  and  $P_2$  will denote the orthogonal projection on  $H_1$  and  $H_2$  respectively.

For every  $u \in H$ ,  $u_1$  and  $u_2$  will denote  $P_1 u$  and  $P_2 u$  respectively.

Since  $G$  is an orthogonal group,  $H_1$  and  $H_2$  are invariant under the action of  $G$ . Using this decomposition of  $H$ , equation (3.2) is equivalent to the following system:

$$(3.4) \quad \begin{cases} P_1 \left( Au - \frac{(Au, u)}{|u|^2} u \right) = 0 \\ P_2 \left( Au - \frac{(Au, u)}{|u|^2} u \right) = 0 \end{cases}$$

where  $|\cdot|$  is the norm in  $H$ .

The operator:

$$Bu = B(u_1, u_2) = P_2 \left[ Au - \frac{(Au, u)}{|u|^2} u \right]$$

is  $G$ -equivariant. In fact:

$$\begin{aligned} B(gu_1, gu_2) &= P_2 \left( A(gu_1, gu_2) - \frac{(Agu, gu)}{|gu|^2} gu \right) = \\ &= P_2 \left( gAu - \frac{(Au, u)}{|u|^2} gu \right) = gP_2 \left[ Au - \frac{(Au, u)}{|u|^2} u \right]. \end{aligned}$$

We have used the fact that  $G$  is orthogonal and  $A$  is  $G$ -equivariant. Moreover  $B$  satisfies all the hypotheses of Lemma 3.1 (see [M]). Then we deduce the existence of a cone  $C = \{u : |P_2 u| < c|P_1 u|\}$ , a number  $\delta > 0$  and a  $C^1$ - $G$ -equivariant function:

$$\phi : \{u_1 \in H_1 : 0 < |u_1| < \delta\} \rightarrow H_2$$

such that:

$$(3.5) \quad \lim_{u_1 \rightarrow 0} \phi'(u_1) = 0$$

$$(3.6) \quad B(u_1, u_2) = 0 \text{ if and only if } u_2 = \phi(u_1).$$

Let us call  $M$  the graph of  $\phi$ . This is a  $G$ -invariant  $n$ -dimensional manifold, where  $n$  is the dimension of  $H_1$ .

Then we consider the sphere  $S_\rho = \{u \in H : |u| = \rho\}$  and prove the following:

**LEMMA 3.2** - There exists  $\rho_0 > 0$ , such that, for each  $0 < \rho < \rho_0$ ,  $M_\rho : S_\rho \cap M$  is  $G$ -homeomorphic to the  $(n-1)$ -dimensional sphere  $S_1$  in  $H_1$ .

**PROOF:** For  $u_1$  in  $S_1$ , i.e.  $|u_1| = 1$ , and  $\varepsilon$  small enough we consider the map:

$$S_1 \times ]0, \varepsilon[ \rightarrow \mathbb{R}$$

defined by:  $g(u_1, t) = \frac{1}{2} |tu_1 + \phi(tu_1)|^2$ . We have:



$$\begin{aligned}\frac{\partial}{\partial t} g(u_1, t) &= (tu_1 + \phi(tu_1), u_1 + \phi'(tu_1)u_1) = \\ &= t(u_1, u_1) + t(u_1, \phi'(tu_1)u_1) + (\phi(tu_1), u_1) + \\ &+ (\phi(tu_1), \phi'(tu_1)u_1) .\end{aligned}$$

Thus by the property of  $\phi$  and  $\phi'$  it follows that

$$\frac{\partial}{\partial t} g(u_1, t) = t|u_1|^2 + o(t) .$$

Therefore there exists  $\rho_0 > 0$  such that

$$\frac{\partial}{\partial t} g(u_1, t) > 0 \text{ for every } t \in (0, \rho_0) .$$

Then we can apply the implicit function theorem to solve the equation:

$$g(u, t) = \frac{1}{2} \rho^2 \text{ (for } \rho < \rho_0)$$

and we can say that there exists a unique function:

$$\tau : S_1 \rightarrow (0, \rho_0)$$

with the property that:

$$g(u_1, t) = \frac{1}{2} \rho^2, \text{ if and only if } t = \tau(u_1) .$$

Note that since  $g$  is  $G$ -invariant, also  $\tau$  is  $G$ -invariant.

Then we consider the map:

$$\begin{aligned}f : S_1 \rightarrow M_\rho \text{ defined by:} \\ f(u_1) = \tau(u_1)u_1 + \phi(\tau(u_1)u_1) .\end{aligned}$$

This map is the required  $G$ -homeomorphism.

It is easy to see that, since  $\tau$  is  $G$ -invariant, and  $\phi$  is  $G$ -equivariant,  $f$  is  $G$ -equivariant. Moreover  $f$  is bijective because if  $u_1 \neq \bar{u}_1$ , then  $\tau(u_1)u_1 \neq \tau(\bar{u}_1)u_1$ , since  $\tau(u_1)$  and  $\tau(\bar{u}_1)$  are both positive, and this implies that the corresponding points in the graph of  $\phi$  are different. □

We now need the following lemma whose proof can be found in [M].

LEMMA 3.3 - Every critical point of  $J|_{M_\rho}$  is a critical point of  $J|_{S_\rho}$ , and this provides a solution of equation (3.2).

Now we can state the main result of this section:

Theorem 3.1 - Let  $H_1$  be the eigenspace of  $A'(0)$  corresponding to an isolated eigenvalue  $\lambda_0$  of finite multiplicity  $n$ . Suppose that  $A$  is  $G$ -equivariant where  $G$  is a finite group and set

$$(3.6') \quad v = \text{G.C.D.}\{|0_u|, u \in S_1 \cap H_1\}.$$

Then if  $v > 2$ , for every  $\rho$  small enough, equation (3.2) has at least  $n$  orbits of solutions  $u$  such that  $\|u\|_{H_1} = \rho$ , provided that every solution is counted with its multiplicity in the sense of Definition 1.1.

PROOF: By lemma 3.3, it is sufficient to estimate the number of critical points of  $J|_{M_\rho}$ .

By lemma 3.2 we know that  $M_\rho$  is  $G$ -diffeomorphic to  $S_1 \cap H_1$  which is a  $(n-1)$  dimensional sphere. Therefore theorem 2.2 can be applied (cf. remark 2.4) with

$$\delta = \text{G.C.D.}\{|0_u| : u \in K\}$$

with

$$K = \{u \in M_\rho \mid (J|_{M_\rho})'(u) = 0\}.$$

By the definition of  $\delta$ ,

$$\delta > \text{G.C.D.}\{|0_u| : u \in M_\rho\}$$

and by lemma 3.2

$$\text{G.C.D.}\{|0_u| : u \in M_\rho\} = v.$$

Then the conclusion follows. □

REMARK 3.2 - Notice that the number  $2$  defined by (3.6') depends only on the action of  $G$  on the eigenspace  $H_1$  corresponding to the eigenvalue  $\lambda_0$  (or more precisely on  $H_1 \cap S_1$ ). The action of the group on  $H_2 = H_1^\perp$  is not relevant to the bifurcation from  $\lambda_0$ .

As an application of the previous Theorem we will study the following problem:

$$(3.7) \quad -\Delta u = \mu f(u), \quad u \in H_0^1(\Omega)$$

where  $\Omega \subset \mathbb{R}^2$  is the square  $(-\frac{\pi}{2}, \frac{\pi}{2})^2$  and  $H_0^1(\Omega)$  denotes the usual Sobolev space.  $\Omega$  is invariant under the action of the group of the symmetry of the square which is denoted by  $C_{4v}$ . To be more precise  $C_{4v}$  is the group of the following matrices:

$$C_{4v} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The group  $C_{4v}$  induces an action on  $H_0^1(\Omega)$  in the following way

$$T_g u = u(g(x, y)) \quad , \quad g \in C_{4v} \quad (x, y) \in \Omega.$$

We suppose that

$$(3.8) \quad f \in C^1(\mathbb{R})$$

$$(3.9) \quad f'(t) < a_1 + a_2 |t|^p \quad p \in \mathbb{N}$$

$$(3.10) \quad f(0) = 0 \quad ; \quad f'(0) = 1.$$

We now set  $F(t) = \int_0^t f(s) ds$ . Then by standard arguments it follows that the functional

$$(3.11) \quad J(u) = \int_{\Omega} F(u) dx dy$$

is a  $C^2 - C_{4v}$ -invariant functional on  $H_0^1(\Omega)$ . On  $H_0^1(\Omega)$  we use the gradient norm:

$$\int_{\Omega} |\nabla u|^2 dx dy$$

$\langle \cdot, \cdot \rangle$  will denote the corresponding scalar product. Then the critical points of  $J$  on the sphere  $\{ \|u\|_{H_1} = \varepsilon \}$  satisfy the following equation

$$\int_{\Omega} f(u) v dx dy = \lambda \int_{\Omega} \nabla u \nabla v dx dy$$

which gives a solution of (3.7) with  $\mu = \frac{1}{\lambda}$ .

Now we define the operator  $A : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  by the formula

$$\langle A(u), v \rangle = \int_{\Omega} f(u) v dx dy \quad \text{for every } v \in H_0^1(\Omega).$$

It is known that, by virtue of (3.8), (3.9) and (3.10)  $A \in C^1(H_0^1(\Omega))$  and

$$\langle A'(0)[v], w \rangle = \int_{\Omega} f'(0) v w dx dy = \int_{\Omega} v w dx dy.$$

The eigenvalue problem for  $A'(0)$ , in weak formulation, takes the form

$$\langle A'(0)[v], w \rangle = \lambda \langle v, w \rangle \quad \forall w \in H_0^1(\Omega)$$

which explicitly gives

$$\int_{\Omega} v w dx dy = \lambda \int_{\Omega} \nabla v \nabla w dx dy \quad \forall w \in H_0^1(\Omega)$$

or

$$-\lambda \Delta v = v.$$

Therefore the eigenvalues of  $A'(0)$  are given by the formula

$$\lambda_{m,n} = \frac{1}{2 + n^2}$$

and the corresponding eigenfunctions are:  $\sin n(x + \frac{\pi}{2}) \sin m(x + \frac{\pi}{2})$ , that is:

$$(3.13) \quad \begin{cases} \sin nx \cdot \sin my & n, m \text{ both even} \\ \sin nx \cdot \cos my & n \text{ even, } m \text{ odd} \\ \cos nx \cdot \sin my & n \text{ odd, } m \text{ even} \\ \cos nx \cdot \cos my & n, m \text{ both odd} \end{cases}$$

It is easy to see that in the first three cases  $C_{4,v}$  induces an action on the eigenspaces corresponding to  $\lambda_{mn}$  which does not have fixed points.

Moreover, since  $|C_{4,v}| = 8$  it is obvious that

$$\delta : \text{G.C.D.}\{C_{4,v}(u) \mid u \in H_1 \cap S^1\} > 2$$

if  $H_1$  is an eigenspace of  $A'(0)$  (and of  $-\Delta$ ) which does not contain fixed points.

Therefore, by theorem 3.1, the following result follows;

Theorem 3.2 - Let  $\mu$  be an eigenvalue of  $-\Delta$  and suppose that  $\mu = m^2 + n^2$  implies that  $m$  and  $n$  are not both odd. Then  $\mu$  is a bifurcation point of (3.7) and if  $k$  is the dimension of the corresponding eigenspace then for every small  $\epsilon$ , (3.7) has at least  $2(k+1)$  solutions  $u$  such that  $\|u\|_{H_0^1(\Omega)} = \epsilon$ .

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ABSTRACT (continued)

We give an estimate on the number of "branches" bifurcating from an eigenvalue of  $A^*(0)$ .



